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A reduction principle for asymptotically triangular differential systems☆

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Abstract

Problems of the stability of non-linear non-autonomous systems of differential equations with a special class of asymptotically vanishing perturbations are considered. The problem of reducing a problem on the stability of the equilibrium of a perturbed system to a problem on stability with respect to a non-linear approximation system which has a triangular form is solved. Applications of the results of the investigations to mechanical systems with a variable mass and time-varying equations of the constraints are presented.

1. Systems with asymptotically vanishing perturbations

Suppose $\mathbb{R}^n$ is an n-dimensional Euclidean space, $||z||$ is the norm of an element z from $\mathbb{R}^n$ and $B^n_\Delta = \{z \in \mathbb{R}^n : ||z|| < \Delta\}$ is a sphere with its centre at the origin of coordinates and a radius $\Delta > 0$.

Definition 1. Suppose U is a neighbourhood of the origin of the coordinates of space $\mathbb{R}^n$ and Y: $(z, t) \to Y(z, t)$ is a continuous function, defined on the set $U \times \mathbb{R}^+$ such that $Y(0, t) = 0$, $\forall t \geq 0$. We shall say that the function Y is *locally asymptotically vanishing* if, for any number $\epsilon > 0$, a compact neighbourhood K of the point $z = 0$ exists, which is contained in the sphere B^n_ϵ, and, for any number $\mu > 0$, on instant of time $t = t(K, \mu) > 0$ exists such that the following condition is satisfied

$$\|Y(z, t)\| \leq \mu, \quad \forall t > t(K, \mu) \tag{1.1}$$

The function $Y(z, t) = \varphi(t)\psi(z)$, where $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $\psi(0) = 0$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with a bounded integral on the semi-axis $\mathbb{R}^+$ serves as an example of a locally asymptotically vanishing function. This type of perturbations of systems of differential equations has most frequently been investigated in the problem of the stability of equilibrium positions.

We will assume that the following pair of systems of differential equations is given

$$\dot{x} = X(x, t) \tag{1.2}$$

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$$\dot{y} = X(y, t) + Y(y, t) \tag{1.3}$$

The functions X and Y are defined in the domain $U \times \mathbb{R}^+$, they are continuous and vanish at the origin of coordinates of the space $\mathbb{R}^n$ for all $t \geq 0$. Furthermore, suppose the function X has continuous and uniformly bounded partial derivatives with respect to the coordinates of the states of the vector x in the domain of the points $(x, t) \in U \times \mathbb{R}^+$. We shall also assume that the function Y does not violate the conditions for the existence and uniqueness of the solutions of system (1.3) and is locally asymptotically vanishing.

In a certain sense, system (1.2) is the limiting system for the perturbed system (1.3) when $t \to +\infty$.

We shall henceforth use the well-known definitions of stability, uniform stability, uniform asymptotic stability and stability in the whole (see Refs. 1–4, for example).

To investigate the problem of the stability of the zero solution of non-linear non-autonomous systems of differential equations under the action of locally asymptotically vanishing perturbations, we shall first prove a theorem on the reduction principle of a general form.

We recall that a Hahn function $\mathbb{R}^+ \to \mathbb{R}^+$ is understood to mean[2] a continuous, strictly monotonically increasing positive function such that $a(0) = 0$.

Theorem 1. *Suppose the following conditions are satisfied*:

1) *the solution $y = 0$ of system (1.3) is uniformly stable*;
2) *the solution $x = 0$ of system (1.2) is uniformly asymptotically stable*;
3) *the function $Y(y, t)$ is locally asymptotically vanishing*.

The zero solution of the perturbed system (1.3) is then uniformly asymptotically stable.

Proof. Since the zero solution of system (1.2) is uniformly asymptotically stable, a number $\Delta > 0$ exists which defines the sphere B_Δ^n of initial states x_0 located in the domain of uniform attraction of the solution $x = 0$ of system (1.2). Moreover, according to Theorem 5.1 in Ref. 1, a continuously differentiable Lyapunov function $V : B_\Delta^n \times \mathbb{R}^+$, Hahn functions a, b and c, and a number $N > 0$ exist for system (1.2) such that the following conditions hold in the domain $B_\Delta^n \times \mathbb{R}^+$

$$a(\|x\|) \leq V(x, t) \leq b(\|x\|), \quad \dot{V}_{(2)}(x, t) \leq -c(\|x\|), \quad \|\partial V(x, t)/\partial x\| \leq N$$

We will now calculate the time derivative $\dot{V}_{(3)}$ of the function $V(y, t)$ by virtue of system (1.3) and estimate the expression for the derivative from above in the domain $B_\Delta^n \times \mathbb{R}^+$. We obtain

$$\dot{V}_{(3)}(y, t) = \dot{V}_{(2)}(y, t) + (\partial V(y, t)/\partial y) Y(y, t) \leq -c(\|y\|) + N\|Y(y, t)\| \tag{1.4}$$

In accordance with the uniform stability of the origin of the coordinates of system (1.3) for any number $\epsilon > 0$, it can be shown that the number $\delta = \delta(\epsilon) > 0$ is such that the solution $y(y_0, t_0, t)$ with the initial data $y_0 \in B_\delta^n$, $t_0 \geq 0$ is contained in the sphere B_ϵ^n for all $t \geq t_0$. Without any loss in generality in the subsequent discussion, we will assume that $\epsilon < \Delta$. □

Suppose a pair of numbers $\mu > 0$ and $\nu = \nu(\mu) > 0$ also corresponds to the definition of the uniform stability of the zero solution of system (1.3), where $\nu < \delta$, that is, the condition

$$y_0 \in B_\nu^n \Rightarrow \|y(x_0, y_0, t_0, t)\| \leq \mu, \quad \forall t_0 \geq 0, \quad \forall t \geq t_0 \tag{1.5}$$

is satisfied.

Since the perturbation Y in system (1.3) is a locally asymptotically vanishing function, then, for a certain number $t_\varepsilon = t(K, \mu) > 0$ (in this case, K is the closure of the sphere B_ϵ^n), the inequality

$$\|Y(z(t), t)\| \leq c(\nu)/2, \quad \forall t > t_\varepsilon$$

will be satisfied.

We will now show that a quantity $T > t_\varepsilon$, which is independent of $t_0 \geq 0$, exists such that the solution $y(t) = y(y_0, t_0, t)$ with an initial state $y_0 \in B_\delta^n$ when $t = T + t_0$ falls in the sphere B_ν^n. But, then, this solution will not leave the sphere B_μ^n

for all $t > T + t_0$, whence the uniform asymptotic stability of the zero solution of system (1.3) will follow. In fact, if this is not so, the values of the function $y(t)$ will belong to a spherical layer $B^n_\delta \backslash B^n_\eta$ for all $t \geq t_\varepsilon$. In this case, the equality

$$V(y(t), t) = V(y(t_\varepsilon), t_\varepsilon) + \int_{t_\epsilon}^{t} \dot{V}_{(3)}(y(s), s)ds$$

can be written for the superposition of the functions $V(y(t), t)$, from which, when $t > t_\varepsilon$ and account is taken of the first condition of the theorem and conditions (1.1), (1.4) and (1.5), the inequality

$$V(y(t), t) \leq b(\mu) - (t - t_\epsilon)c(\nu) + N\int_{t_\epsilon}^{t} \|Y(y(s), s)\| ds \tag{1.6}$$

follows.

On the basis of relation (1.1), it can also be stated that a number $T_1 = T_1(\epsilon) > 0$ exists such that

$$\int_{t_\epsilon}^{t} \|Y(y(s), s)\| ds < \frac{1}{2}c(\nu)(t - t_\epsilon)$$

only if $t > T_1 + t_\epsilon$.

We now fix the positive number T by subjecting it to the condition

$$T > T_1 + t_\epsilon + b(\mu)/c(\nu) \tag{1.7}$$

It is clear that the quantity $T > 0$ chosen in this way is independent of both the initial instant of time $t_0 \geq 0$ and of the initial states $t_0 \in B^n_\delta$. By construction, when $t > T$, the right-hand side of equality (1.6), by virtue of condition (1.7), is a negative quantity which, in the spherical layer $B^n_\epsilon \backslash B^n_\eta$, contradicts the sign of the left-hand side of these inequalities. Hence, it has been shown that the solution $y(t)$ remains in the sphere B^n_μ for all $t > T$. It also follows from this, by virtue of the arbitrariness of the choice of the number $\mu > 0$, that the zero solution of system (1.2) is uniformly asymptotically stable.

We will now consider a theorem concerning stability in the large. In order to shorten the formulation of the corresponding assertion, we will introduce the following concept which appears in different modifications in Refs. 4–14.

Suppose the domain of definition of the states of system (1.2) is identical to the whole of phase space, that is, $U = \mathbb{R}^n$. We shall say that the zero solution of system (1.2) is *uniformly stable in the large* if it is uniformly stable and, moreover, the limiting equality for the solutions $x(x_0, t_0, t)$ of system (1.2)

$$\lim_{t \to +\infty} \|x(x_0, t_0, t)\| = 0 \tag{1.8}$$

is uniformly satisfied with respect to $t_0 \geq 0$ and $x_0 \in K$ for each compact neighbourhood K of the origin of coordinates of the space $\mathbb{R}^n$.

We shall also say that the solutions of system (1.2) are uniformly bounded when $t \to +\infty$ if, for any compact neighbourhood $K \subset \mathbb{R}^n$ of the origin of coordinates, it is possible to find a positive number M such that

$$\|x(x_0, t_0, t)\| \leq M, \quad \forall x_0 \in K, \quad \forall t_0 \geq 0, \quad \forall t \geq t_0$$

Definition 2. Suppose Y: $(z, t) \to Y(z, t)$ is a continuous function, defined in the set $\mathbb{R}^n \times \mathbb{R}^+$ such that $Y(0, t) = 0$, $\forall t \geq 0$. We shall say that Y is an *asymptotically vanishing function* if, for any compact neighbourhood K of the point $z = 0$ and any number $\mu > 0$, an instant of time $t = t(K, \mu) > 0$ exists such that condition (1.1) is satisfied.

Theorem 2. *Suppose $U = \mathbb{R}^n$ and the following conditions are satisfied*:

1) *the solution $y = 0$ of system is uniformly stable*;
2) *the solution $x = 0$ of system (1.2) is uniformly stable in the large*;
3) *the function $Y(y, t)$ is an asymptotically vanishing function*;
4) *the solutions of system (1.3) are uniformly bounded when $t \to +\infty$.*

The zero solution of system (1.3) is then uniformly stable in the large.

Proof. By Theorem 1, satisfaction of the conditions of this theorem implies the uniform asymptotic stability of the zero solution of system (1.3). Therefore, in order to prove Theorem 2, it suffices to show that every solution $y(y_0, t_0, t)$ with arbitrary initial data $y_0 \in \mathbb{R}^n$ tends to zero when $t \to +\infty$. On account of this, we use the last condition of Theorem 2 which guarantees the existence of a number $M > 0$ such that

$$\|y(y_0, t_0, t)\| \le M, \quad \forall t \ge t_0 \tag{1.9}$$

For brevity, we will only present the scheme for the subsequent proof of this theorem which can be constructed using a repetition of the steps in the proof of Theorem 1. To do this, it is first necessary to use estimate (1.9) instead of inequality (1.5) and, secondly, instead of the function V, which satisfies the conditions of Theorem 1, it is necessary to use the corresponding Lyapunov function, which satisfies the conditions of Theorem 16.3 in Ref. 3 (the case of stability in the large). Thirdly, it is necessary to use Definition 2 instead of Definition 1. Then, in the subsequent reasoning, the structure of the proof of Theorem 1 can be fully utilized, apart from an unimportant change which also leads to the satisfaction of the limiting equality

$$\lim_{t \to +\infty} \|y(y_0, t_0, t)\| = 0$$

2. Asymptotic triangular systems

We will assume that system (1.2) has the form

$$\dot{x} = f(x, y, t), \quad x \in U_x; \quad \dot{y} = g(y, t), \quad y \in U_y; \quad t \ge 0 \tag{2.1}$$

where $U = (U_x, U_y)$ is an open connected neighbourhood of the origin of coordinates of the Euclidean space $\mathbb{R}^{p+q}$ and, moreover, $U_x \subset \mathbb{R}^p$, $U_y \subset \mathbb{R}^q$.

For each initial state $z_0 = (x_0, y_0)$ from U and the initial instant $t_0 \ge 0$, we shall denote the solution of system (1.1) with the initial conditions $z(z_0, t_0, t_0) = z_0 = (x_0, y_0)$ by

$$z(z_0, t_0, t) = (x(z_0, t_0, t), y(z_0, t_0, t))$$

Suppose the perturbed system (1.3) has the form

$$\dot{x} = f(x, y, t) + Y_1(x, y, t), \quad x \in U_x; \quad \dot{y} = g(y, t) + Y_2(x, y, t), \quad y \in U_y; \quad t \ge 0 \tag{2.2}$$

As before, the function $Y = (Y_1, Y_2)$ plays the role of the perturbed system (2.1).

We now introduce the following type of perturbations of the differential systems.

Definition 3. We shall say that system (2.2) is *locally asymptotically triangular* (*asymptotically triangular*) if the function Y is a locally asymptotic vanishing (asymptotically vanishing respectively) function.

Note that the problem of the stability of the zero solution of triangular systems of differential equations (system (2.1)) has been considered by many authors (see Refs. 6–11,15–19).[a]

Autonomous systems have been investigated in Refs. 6–8,16–18 and non-autonomous systems were investigated in Refs. 9,10,15,16,19. Two approaches have been mainly used to solve the problem of the stability of the zero solution: the direct Lyapunov method (Refs. 5,10,15–17,19)[a] (sign-definite Lyapunov functions (Refs. 15,17,19) or constant-sign Lyapunov functions (Refs. 5,10,16)[a] and an approach associated with the Florio Seibert problem,[20] the source of which is a problem on partial stability[12] and the reduction principle.[13] This method was developed[14] in a general form for problems of topological dynamics as applied to the problem of the relative stability of invariant sets.

The main conclusion reached in Refs. 6–11,15–19[a] is the fact that, if the solution $x = 0$ of the system

$$\dot{x} = f(x, 0, t), \quad x \in \mathbb{R}^n, \quad t \ge 0 \tag{2.3}$$

[a] See also: *Kalitin BS*, The construction of constant-sign Lyapunov functions. Minsk, 1979. Deposited at VINITI 17.05.79. No. 1760–79.

is uniformly asymptotically stable and the solution $y=0$ of the system

$$\dot{y} = g(y, t), \quad y \in \mathbb{R}^q, \quad t \geq 0 \tag{2.4}$$

is uniformly asymptotically stable (or uniformly stable), then the origin of the coordinates of system (1.1) is uniformly asymptotically stable (uniformly stable respectively).

We shall prove the reduction principle for the asymptotically triangular systems of non-autonomous differential equations (2.2).

Theorem 3. *We will assume that the following conditions are satisfied*:

1) *the origin of coordinates of system* (2.2) *is uniformly stable with respect to* y;
2) *the zero solutions of system* (2.3) *and system* (2.4) *are uniformly asymptotically stable*.

The origin of coordinates of the asymptotically triangular system (2.2) *is then uniformly asymptotically stable*.

Proof. By virtue of the stability with respect to the coordinates of the vector y for any number $\mu > 0$ ($B^q_\mu \subset U_y$), a number $\nu = \nu(\mu) > 0$ exists such that, for any initial constant of time $t_0 \geq 0$ and for any initial state $(x_0, y_0) \in B^{p+q}_\nu$, we shall have

$$\|y(t, t_0, x_0, y_0)\| < \mu, \quad \forall t \geq t_0 \tag{2.5}$$

Since the zero solution of system (2.3) is uniformly asymptotically stable, then, by Theorem 5.1 in Ref. 1, a number $\rho > 0$ ($B^p_\rho \subset U_x$), a continuously differentiable function $\upsilon : B^p_\rho \times \mathbb{R}^+ \to \mathbb{R}^+$, Hahn functions a, b and c and a number $N > 0$ exist such that the conditions

$$a(\|x\|) \leq \upsilon(x, t) \leq b(\|x\|), \quad \dot{\upsilon}_{(4)}(x, t) \leq -c(\|x\|), \quad \|\partial\upsilon(x, t)/\partial x\| \leq N$$

hold in the domain $B^p_\rho \times \mathbb{R}^+$. □

By assumption, the right-hand sides of system (2.2) satisfy the Lipschits condition, and a number $L = L(\rho) > 0$ can therefore be found for which the following inequality holds

$$\|f(x, y, t) + Y_1(x, y, t) - f(x, \tilde{y}, t) - Y_1(x, \tilde{y}, t)\| \leq L\|y - \tilde{y}\|, \quad \forall x \in B^p_\rho, \quad \forall y, \tilde{y} \in B^q_\rho, \\ \forall t \geq 0 \tag{2.6}$$

We fix the number $\varepsilon > 0$ in an arbitrary manner. Without loss of generality in the subsequent reasoning, it can obviously be assumed that

$$\epsilon < \rho \tag{2.7}$$

We choose the number $\gamma = \gamma(\varepsilon) > 0$ such that

$$b(\gamma) < a(\epsilon), \quad \gamma < \epsilon \tag{2.8}$$

and the number ν is so small that condition (2.5) and the following inequalities are simultaneously satisfied

$$\nu < \gamma, \quad \mu < \min\left\{\frac{\sigma}{NL}, \frac{\rho}{2}\right\}; \quad \sigma = -\sup_{\gamma \leq \|x\| \leq \epsilon,\, t \geq 0} \dot{\upsilon}_{(4)}(x, t) \tag{2.9}$$

We now immediately proceed to the proof of the uniform stability of the zero solution of system (2.2). To do this, we will show that, if $z_0 \in B^{p+q}_\nu$ and $t_0 \geq 0$, then the solution $z(z_0, t_0, t) = (x(t), y(t))$ of system (2.2) does not leave the sphere B^{p+q}_ϵ for all $t_0 \geq 0$. Actually, since, according to Condition 1 of Theorem 1, the zero solution of this system is stable with respect to the coordinates of the vector $y(t)$, it is sufficient to show that the inequality

$$\|x(t)\| \leq \epsilon, \quad \forall t \geq t_0 \tag{2.10}$$

is satisfied for the components of the vector $x(t)$.

If, conversely, the condition is violated, then it is obvious that instants of time $t_2 > t_1 > t_0$ exist such that

$$\|x(t_1)\| = \gamma, \quad \gamma < \|x(t)\| < \epsilon \ \text{ when } \ t_1 < t < t_2 \ \text{ и } \ \|x(t_2)\| = \epsilon$$

We now write the time derivative $\dot{v}_{(2)}$, calculated from the function $v(x, t)$ by virtue of system (2.2) (a prime denotes transposition)

$$\dot{v}_{(2)}(x, y, t) = \dot{v}_{(4)}(x, t) + (\partial v(x, t)/\partial x)'(f(x, y, t) + Y_1(x, y, t) - f(x, 0, t) - Y_1(x, 0, t))$$

From this, using the Lipschits inequality (2.6) and, also, relations (2.5) and (2.9), we obtain the limits

$$\dot{v}_{(2)}(x, y, t) = \dot{v}_{(4)}(x, t) + NL\|y(t)\| \le -\sigma + NL\mu < 0 \text{ for all } t_1 \le t \le t_2$$

Hence, in the interval $[t_1, t_2]$, the functions $v(x(t), t)$ decay strictly monotonically, and therefore, when account is taken of the conditions which they satisfy, we shall have the relations

$$a(\epsilon) = a(\|x(t_2)\|) \le v(x(t_2), t_2) < v(x(t_1), t_1) \le b(\|x(t_1)\|) = b(\gamma)$$

However, this contradicts inequality (2.8). Consequently, the inverse assertion holds, and we also obtain uniform stability with respect to the coordinates of the vector $x(t)$.

The construction of the proof of Theorem 1 also enables us to formulate an assertion concerning non-asymptotic stability without any particular difficulty.

Theorem 4. *We will assume that the following conditions are satisfied*:

1) *the origin of coordinates of system* (2.2) *is stable with respect to* y;
2) *the zero solution of system* (2.1) *is uniformly asymptotically stable*.

Proof. Unlike in the proof of Theorem 1, it suffices here to make use of Theorem 13.1 from Ref. 3 which, in the case of system (2.2), guarantees the existence of a constant-sign Lyapunov function (positive-definite with respect to the coordinates of the vector y) and then to apply Theorem 2 from Ref. 10 with the condition that $\varphi(x, t) \equiv 0$. □

We now pass to the last of the proposed results, a theorem on stability in the large.

Theorem 5. *Suppose* $U = \mathbb{R}^{p+q}$. *We assume that system* (2.2) *is asymptotically triangular and satisfies the following conditions*:

1) *the origin of coordinates of system* (2.2) *is uniformly stable with respect to* y;
2) *the zero solution of system* (2.3) *is uniformly stable in the large*;
3) *the zero solution of system* (2.4) *is uniformly stable in the large*;
4) *every solution of system* (2.2) *is bounded when* $t \to +\infty$.

The origin of coordinates of system (2.2) *is then uniformly stable in the large*.

Proof. According to Theorem 1, satisfaction of the first three conditions of Theorem 3 implies the uniform asymptotic stability of the zero solution of system (2.2). But, then, by Theorem 2 the required confirmation also follows. □

3. Examples of the application of the results on stability for the equations of motion of mechanical systems

We will now consider the motion of a constrained material system consisting of n points with variable masses $m(t) = (m_1(t), m_2(t), \ldots, m_n(t))$, the positions of which are defined by the coordinates $x_1(t), x_2(t), \ldots, x_{3n}(t)$. Suppose that the system is subjected to ideal holonomic retaining constraints which are expressed by the system of equations

$$S_i(x_1, x_2, \ldots, x_{3n}, t) = 0, \quad i = 1, 2, \ldots, k \tag{3.1}$$

We shall assume that Eq. (3.1) allow the coordinates of the vector x to be expressed in terms of $m = 3n - k$ independent generalized coordinates $q_1, q_2, \ldots, q_m$. In this case, when there are no reactive forces, the motion of the system is described by the equations[21]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial \Pi}{\partial q_i} + Q_i, \quad i = 1, 2, \ldots, m \tag{3.2}$$

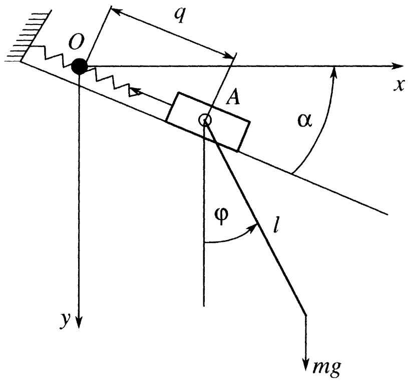

Fig. 1.

where $T = T(q, \dot{q}, t)$, $\Pi = \Pi(q, m(t), t)$; $Q_i = Q_i(q, \dot{q}, t)$ are the non-potential generalized forces acting on the points of the system. Suppose the equilibrium of the system

$$q = 0, \quad \dot{q} = 0 \tag{3.3}$$

is ensured by satisfying the equalities $\partial\Pi/\partial q = 0$ and $Q = 0$.

The equations of motion (3.2) are obviously reduced to the normal form. As a result, we obtain a certain system of differential equations (a system of the form (1.3)) in which the function $Y(y, t)$ reflects the specific changes with time in the masses $m(t)$ and the equations of the constraints (3.1). In this formulation, the problem of the stability of equilibrium (3.3) can be studied using the proposed method for investigating systems with asymptotically vanishing perturbations. In order to confirm this, we will consider an actual situation.

Example 1. Suppose there is a system consisting of a body A of mass M and a mathematical pendulum which is attached to it by a thread of length l (the mass of which we neglect). We assume that a spring with stiffness c is attached to the body A and that the other end of this spring is clamped at a fixed point. The body A can move along an inclined plane at an angle α to the horizontal (Fig. 1) (φ is the angle of inclination of the pendulum from the vertical and q is the magnitude of the displacement of the body A along the inclined plane). We shall assume the the parameters of the system M, m, α and l are continuously differentiable, non-negative functions of time t and that the domains of the values of these parameters are subject to the inequalities

$$0 < M_0 \le M(t) \le M_1, \quad 0 < m(t) \le m_1, \quad 0 < l_0 \le l(t) \le l_1, \quad 0 < \alpha(t) < \pi/2 \tag{3.4}$$

The first two inequalities of (3.4) denote that we are dealing with a system of variable mass here and the last two of them correspond to the changing conditions of the connections between the point masses.

We now set up the differential equations of motion of the system, placing the origin of coordinates on the descending vertical Oy at the point of static equilibrium O. This system has two degrees of freedom. As generalized coordinates, we choose the angle φ of inclination of the pendulum from the vertical and the displacement q of the body A along the inclined plane, measured from the point of static equilibrium.

We shall assume that a change in the masses $M(t)$ and $m(t)$ is not accompanied by corresponding reactive forces. In this case, the force of the action of the spring has a magnitude $F = c|q + \mu(t)|$, where $\mu(t)$ is the static elongation of the spring, which is given by the formula

$$\mu(t) = (M(t) + m(t))g\sin\alpha(t)/c, \tag{3.5}$$

Besides the action of the force of gravity $m\mathbf{g}$, we shall also take account of the force of resistance to the motion of the body A, which is proportional to the velocity $\dot{q}$ and the force of resistance to the rotation of the pendulum which is proportional to its momentum $m(t)l(t)\dot{\varphi}$.

With these assumptions, the equations of motion in the chosen generalized coordinates (q, φ) have the form

$$\begin{aligned}&\frac{d}{dt}((M+m)\dot{q}+ml\cos(\varphi+\alpha)) = -cq-k\dot{q}\\&\frac{d}{dt}((ml^2\dot{\varphi}+m\dot{q}l\cos(\varphi+\alpha))+ml\dot{q}\dot{\varphi}\sin(\varphi+\alpha)) = -gml\sin\varphi-hml\dot{\varphi}\end{aligned} \tag{3.6}$$

where k and hm are the coefficients of the resistance forces. For brevity, we will henceforth omit the argument t in the case of the functions $M(t)$, $m(t)$, $l(t)$ and $\alpha(t)$.

Calculating the total time derivatives and solving Eq. (3.6) for $\ddot{\varphi}$ and $\ddot{q}$, we arrive at the system

$$\begin{aligned}&(\Delta/l)\ddot{q} = -\dot{q}(\dot{M}+\dot{m}+k)-\dot{\varphi}\Pi+ml\dot{\varphi}(\dot{\varphi}+\dot{\alpha})\sin(\varphi+\alpha)-cq+\dot{q}\Pi\cos(\varphi+\alpha)+\\&+(\dot{\varphi}(l\dot{m}+2ml+hm)-m\dot{q}\dot{\alpha}\sin(\varphi+\alpha)+m\sin\varphi)\cos(\varphi+\alpha)\\&\Delta\ddot{\varphi} = (\dot{q}(\dot{M}+\dot{m}+k)+\dot{\varphi}(\Pi-ml(\dot{\varphi}+\dot{\alpha})\sin(\varphi+\alpha))+cq)\cos(\varphi+\alpha)-\\&-(M+m)(\dot{\varphi}(l\dot{m}/m+2\dot{l}+h)+(\dot{m}/m+\dot{l}/l)\dot{q}\cos(\varphi+\alpha)-\dot{q}\dot{\alpha}\sin(\varphi+\alpha)+g\sin\varphi)\end{aligned}$$

where

$$\Delta = l(M+m\sin^2(\varphi+\alpha)),\quad \Pi = (\dot{m}l+m\dot{l})\cos(\varphi+\alpha)$$

We will assume that a change in mass occurs when the following conditions are satisfied

$$m\to 0,\quad \dot{m}\to 0\ \text{ when }\ t\to 0\ \text{ and } |\dot{m}/m|\le m_0,\quad \forall t\ge 0 \tag{3.7}$$

The equations of the first approximation (equations of the type (2.1)) can then be chosen in the form

$$M\ddot{q} = -(\dot{M}+k)\dot{q}-cq \tag{3.8}$$

$$l^2\ddot{\varphi} = -l(2\dot{l}+h)\dot{\varphi}-gl\sin\varphi+\dot{q}((\dot{M}+k)+(M+m)\dot{l}/l)\cos(\varphi+\alpha) \tag{3.9}$$

The conditions for the stability of the equilibrium can be derived separately for two cases: when the slip plane of the body A is fixed, that is,

$$\alpha(t) = \dot{\alpha}_0 = \text{const} \tag{3.10}$$

and when it can move while preserving condition (3.4). We will only consider the first case, which corresponds to the possibility of choosing of an asymptotically triangular system (system (3.8), (3.9) is triangular).

It can be seen that, when equality (3.10) is satisfied, the uniform asymptotic stability of the solution $q=\dot{q}=0$ of Eq. (3.8) can be ensured by the condition

$$\dot{M}(t)+k\ge\gamma>0,\quad \forall t\ge 0 \tag{3.11}$$

which means that, in the case of the condition for a reduction in the mass M of the body A, the modulus of its contraction velocity must not be greater than the coefficient k of the force of resistance to the movement of the body along the slip plane, which is arranged at an invariant angle α_0 to the horizontal. If, however, the mass of the body does not decrease, then inequality (3.11) will be automatically satisfied.

When condition (3.11) is satisfied, the uniform asymptotic stability of the equilibrium $\varphi=\dot{\varphi}=0$ of Eq. (3.9) is equivalent[10] to the uniform asymptotic stability of the solution $\varphi=\dot{\varphi}=0$ of the reduced equation

$$l(t)\ddot{\varphi} = \chi(t)l\dot{\varphi}-\sin\varphi;\quad \chi(t) = -2\dot{l}(t)/l(t)-h \tag{3.12}$$

In the case of this equation, the conditions for uniform asymptotic stability (in assumption (3.7)) can be taken in the form of the inequality

$$\chi(t)\le\delta<0,\quad \forall t\ge 0$$

or, after integration within the limits from 0 to t, in the form

$$l^2(t)\ge l^2(0)\exp(-(h+\delta)t) \tag{3.13}$$

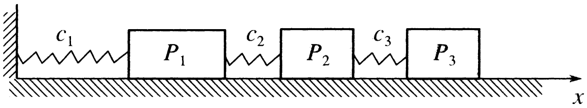

Fig. 2.

From a physical point of view, condition (3.13) imposes a constraint on the change in the length of the pendulum l. This constraint is determined by the coefficient h of the force of resistance to the rotation of the pendulum.

The solution $\varphi = \dot\varphi = 0, q = \dot q = 0$ of the mechanical system being investigated will therefore be uniformly asymptotically stable if conditions (3.4), (3.7), (3.1), (3.11) and (3.13) are satisfied.

Note that, in the situation considered, the stability conditions which have been obtained are independent of the position of the slip plane of the body A, that is, of the angle $\alpha_0 \in]0, \pi/2[$.

Example 2. Consider a mechanical system in the form of three objects (point masses) P_1, P_2, P_3 with corresponding masses m_1, m_2, m_3 arranged on a straight line (Fig. 2). The objects of the system are joined to one another by springs stiffness c_1, c_2, c_3 respectively and, moreover, the left-hand end of the spring P_1 is rigidly clamped. We shall assume that the mass m_1 and the stiffness paramaters c_1, c_2, are continuous functions of time t with continuous derivatives $\dot m_1, \dot c_1$. They satisfy the following conditions

$$\begin{aligned} &m_1 \to 0, \quad \dot m_1 \to 0, \quad c_1 \to 0, \quad \frac{c_2}{m_1} \to 0 \ \text{ when } \ t \to +\infty \\ &\left|\frac{\dot m}{m_1}\right| \le N_1 < +\infty, \quad 0 < \sigma \le \left|\frac{c_1}{m_1}\right| \le N_2 < +\infty, \quad \forall t \ge 0 \end{aligned} \tag{3.14}$$

In particular, it follows from these conditions that $c_2 \to 0$ when $t \to +\infty$. Furthermore, we assume that the object of variable mass P_1 is subject to a resistance force equal to $m_1F(x_1, t)$, which is determined by the continuous function $F(x_1, t)$ and that the point masses P_2 and P_3 are subject to the action of the resistance forces $-k_2\dot x_2$ and $-k_3\dot x_3$, where k_2 and k_3 are positive constants. Subject to these conditions, the equations of motion of the system (in the special case when the masses are constant and there are no resistance forces, see Ref. 22, p. 70, for example) are written in the form

$$m_1(t)\ddot x_1 - \dot m_1(t)\dot x_1 + c_1(t)x_1 - c_2(t)(x_2 - x_1) = -m_1(t)F(x_1, t)\dot x_1 \tag{3.15}$$

$$m_2\ddot x_2 + c_2(t)(x_2 - x_1) - c_3r(x_3 - x_2) = -k_2\dot x_2, \quad m_3\ddot x_3 + c_3(x_3 - x_2) = -k_3\dot x_3 \tag{3.16}$$

where x_1, x_2 and x_3 are generalized coordinates which are equal, respectively, to the distances of the points P_1, P_2 and P_3 from those positions at which the springs are in stress-free state. Here, the equation

$$\ddot x_1 + \left(F(x_1, t) - \frac{\dot m_1(t)}{m_1(t)}\right)\dot x_1(t) + \frac{\dot c_1(t)}{m_1(t)}x_1 = 0 \tag{3.17}$$

and the two equations (3.16) can be taken as the corresponding triangular system (system (2.1)). It can be seen that the difference between the initial system (3.15), (3.16) and the triangular system (3.16), (3.17) is determined by asymptotically vanishing functions.

In the case of Eq. (3.17) (this is Liénard's equation), the conditions for the uniform asymptotic stability of the solution, obtained by the method of limiting equations, were presented in Ref. 23. We continue these investigations by considering the additional problem of stability in the large. We take the Lyapunov function

$$V(x_1, \dot x_1, t) = \frac{1}{2}x_1^2 + \frac{m_1(t)}{2c_1(t)}\dot x_1^2 \tag{3.18}$$

By virtue of condition (3.14), it is positive-definite and infinitely large. By virtue of Eq. (3.17), its time derivative has the form

$$\dot V(x_1, \dot x_1, t) = -\frac{m_1(t)}{c_1(t)}\Psi(x_1, t)\dot x_1^2; \quad \Psi(x_1, t) = F(x_1, t) - \frac{\dot m_1(t)}{m_1(t)} + \frac{m_1(t)}{2c_1(t)}\frac{d}{dt}\left(\frac{c_1(t)}{m_1(t)}\right) \tag{3.19}$$

We require that the inequality

$$\Psi(x_1, t) \geq \Delta > 0, \quad \forall t \geq 0 \tag{3.20}$$

where Δ is a certain constant, should be satisfied. Then, all the solution of Eq. (3.17) will be bounded on the semi-axis $t \geq 0$. When conditions (3.14) and (3.18) are satisfied, we therefore conclude (Ref. 10, Theorem 4) that the equilibrium is uniformly stable in the large.

The second group of equalities from the triangular system which has been presented, that is, Eq. (3.16), is a system with constant coefficients. Consequently, for these equations, the condition for the uniform stability in the large of the zero solution

$$x_2 = \dot{x}_2 = x_3 = \dot{x}_3 = 0$$

can be written, according to the Hurwitz criterion, as

$$\alpha\beta((\alpha+\beta)(\alpha B + \beta A) + (A - B)^2) > AB(\alpha+\beta)^2; \ \ \alpha = \frac{k_2}{m_2}, \ \ \beta = \frac{k_3}{m_3}, \ \ A = \frac{c_3}{m_2}, \ \ B = \frac{c_3}{m_3} \tag{3.21}$$

On the basis of Theorem 5, the equilibrium of the mechanical system considered is therefore uniformly stable in the large if conditions (3.14), (3.20) and (3.21) are satisfied.

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